International Journal of Engineering Applied Sciences and Technology, 2022
Vol. 7, Issue 8, ISSN No. 2455-2143, Pages 102-107
Published Online December 2022 in IJEAST (http://www.ijeast.com)

DYNAMIC AND STABILITY ANALYSIS OF AN INCLINED BEAM SUBJECTED TO MOVING CONCENTRATED LOAD

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Abstract— **The aim of this paper is to develop the dynamical and stability analysis of an inclined beam subjected to a moving concentrated load. The problem is solved for using the method of Laplace transformation for an initial-boundary-value problem, such that an integro-differential solution is obtained and used to simply deal with the condition of singularity by the load functional. The stability of general motion of the elastic system is determined by a direct variational approach. The result derived showed good agreement with that reported in literature.**

Keywords—**Dynamic analysis, stability analysis, inclined beam, moving load, integro-differential problem**

I. INTRODUCTION

A large margin of works in literatures focuses on moving loads on horizontal bridges and foundations [1-10], to mention a few. It may be arguably stated that the intuition and perspective of moving load models on inclined structures elucidated in recent years assume to be a matter of concern in structural dynamics and stability. The growing complexity of mounting infrastructures, such as rails, bridges, and pipelines on uneven terrain to be subjected to moving load makes this research domain highly valid for engineers and mathematicians that are interested in formulating computational structural and load model and developing effective solutions. A good report on the application of moving load on inclined structures is presented in [11], and hence, research becomes a basic modality to idealize our understanding on the performance mechanism of the system. Wu [12] presented the dynamic response of an inclined beam with attention on centrifugal and coriolis fo. Mamandi and Kargarnovin [13] thereafter presented a nonlinear dynamic response of a beam given the effect of transverse shear deformation of the beam. Yang and Wang [14] considered the dynamic and stability of the inclined beam in the context of an axially compressed load using the finite element method. In [15], they further provided insights onto the axial load effect on the beam stiffness based on a semi-analytical solution.

This paper tends to profer a simplistic integro-differential approach in [3] and using a direct variation method to solve for the inclined Euler beam. The approach takes advantage of the so-called extended Galerkin's method to highlight the intrinsic property of the inclined beam while applying the transform method to reduce the physical system onto a green's functional in the context of the moving concentrated load components.

II. THEORETICAL FORMULATION

The following differential equation, and the initial and boundary conditions govern the flexural motion of the inclined beam [14,15]

$$Dw^{IIII} + P_a\left[1 - H\left(x - vt\right)\right]w^{II} + c\dot{w} + \mu\ddot{w} = \left(P_t + P_a w^{I}\right)\delta\left(x - vt\right) = P\delta\left(x - vt\right) \tag{2.1}$$

$$w\left(x,0\right) = \dot{w}\left(x,0\right) = 0, \quad w\left(0,t\right) = w^{I}\left(0,t\right) = w\left(l,t\right) = w^{I}\left(l,t\right) = 0. \tag{2.2}$$

The inclined beam having a moving concentrated load, $\mathbf{P_0}$is presented in Figure 1, **m** denotes the mass per unit length, **E** the Young's modulus, **I** the second moment of area, **c** the damping coefficient, **w** is the transverse deflection with respect to the inclined beam, the loads: $\mathbf{P_t}$ and $\mathbf{P_a}$ are the transverse and axial load component along the beam.

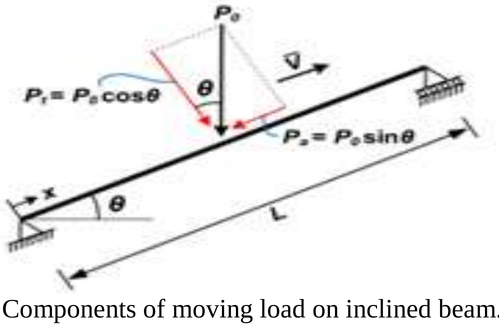

Figure 1. Components of moving load on inclined beam.

Adopted from [14]

By Laplace transformation

$$\bar{w} = \int_0^\infty e^{-st} w(x,t)\, dt, \quad \bar{K} = \int_0^\infty e^{-st} K(x,t)\, dt \qquad (2.3)$$

we get:

$$D\bar{w}^{IIII} + P_a \left[1 - H\left(x - vt\right)\right] \bar{w}^{II} + cs\bar{w} + \mu s^2 \bar{w} = \bar{K} \qquad (2.4)$$

observing that $c = 2\omega_i \mu \xi_i$ represent the i^th modal damping ratio of the vibrating system

$$D\bar{w}^{IIII} + P_a \left[1 - H\left(x - vt\right)\right] \bar{w}^{II} + 2\omega_i \mu \xi_i s\bar{w} + \mu s^2 \bar{w} = \left(\bar{P}_t + \bar{P}_a w^I\right) \delta\left(x - vt\right) \qquad (2.5)$$

In order to obtain $\bar{w}$ from (2.5), let the following series expansions be used

$$\bar{w}(s,x) = \sum_i A_i\left(s\right) W_i\left(x\right), \quad \bar{K}\left(s,x\right) = \sum_i \mu B_i\left(s\right) W_i\left(x\right) \qquad (2.6)$$

Eigenvalues and Eigenvectors –
The coordinate functions are chosen as the eigenfunctions of the self-adjoint auxiliary problem as

$$DW_i^{IIII} + P_a \left[1 - H\left(x - vt\right)\right] W_i^{II} - \mu \omega_i^2 W_i = 0 \qquad (2.7)$$

$$U\left[W_i\left(x\right)\right]_B = 0 \qquad (2.8)$$

It follows simply from (2.7-2.8) that the function in

$$W_i\left(x\right) = \sum_{j=1}^{\infty} \alpha_{ij} Y_j\left(x\right) = \sum_{j=1}^{\infty} \alpha_{ij} \sin\frac{j\pi x}{l} \qquad (2.9)$$

The function $Y_j\left(x\right)$ thus satisfy the boundary conditions. The solution maybe expressed in the so-called extended Galerkin's approach, and observing orthogonality condition so that we arrive at

$$\sum_{j=1}^{\infty} \alpha_{ij} \left\{ \begin{bmatrix} D\left(\frac{j\pi}{l}\right)^4 + P_a\left(\frac{j\pi}{l}\right)^2 \times \\ \left[\frac{1}{2j\pi}\left(\sin\frac{2j\pi vt}{l} - \sin 2j\pi\right) - \frac{vt}{l}\right] - \mu\omega_i^2 \end{bmatrix} \delta_{jk} + P_a\left(\frac{j\pi}{l}\right)^2 \times \begin{bmatrix} \frac{2\sin\pi\left(j-k\right) + 2\sin \pi vt/l\left(j-k\right)}{2\pi\left(j-k\right)} - \\ \frac{2\sin\pi\left(j+k\right) + 2\sin \pi vt/l\left(j+k\right)}{2\pi\left(j+k\right)} \end{bmatrix} \right\} = 0 \qquad (2.10)$$

noting that

$$H\left(x - vt\right) = \begin{cases} 0 & for \quad x \le vt \\ 1 & for \quad x > vt \end{cases} \qquad (2.11)$$

put:

$$\Pi_{jk} = \begin{bmatrix} D\left(\frac{j\pi}{l}\right)^4 + P_a\left(\frac{j\pi}{l}\right)^2 \times \\ \left[\frac{1}{2j\pi}\left(\sin\frac{2j\pi vt}{l} - \sin 2j\pi\right) - \frac{vt}{l}\right] - \mu\omega_i^2 \end{bmatrix} \delta_{jk} + P_a\left(\frac{j\pi}{l}\right)^2 \times \begin{bmatrix} \frac{2\sin\pi\left(j-k\right) + 2\sin \pi vt/l\left(j-k\right)}{2\pi\left(j-k\right)} - \\ \frac{2\sin\pi\left(j+k\right) + 2\sin \pi vt/l\left(j+k\right)}{2\pi\left(j+k\right)} \end{bmatrix} = 0 \qquad (2.12)$$

The eigenvalues for the problem may be solved for if we attempt to find the determinant of Π_{jk}

$$\det\left|\Pi_{jk}\right| = 0 \qquad (2.13)$$

By back substitution of eigenvalues in (2.10), the coefficients are obtained. We see from (2.7-2.8) that the coordinate functions satisfy the following orthonormality conditions; assuming that the weight of the moving mass is far negligible when compared to that of the beam

$$\int_0^1 \mu Y_j\left(\phi\right) Y_k\left(\phi\right) dx = \delta_{jk} \qquad (2.14)$$

One must realize that the nature of operator A in (2.6) may result to an expression for a complicated integro-differential equation. Therefore, in order to keep things simple here we set (2.6) in (2.5) as

$$\sum_i A_i \left\{ DW_i^{IIII} + P_a\left[1 - H\left(x - vt\right)\right] W_i^{II} + \mu s^2 W_i \right\} = \sum_i \mu\left(B_i - 2\omega\xi s\right) W_i \qquad (2.15)$$

Taking note of (2.7) and (2.15)

$$\sum_i A_i \mu\left(\omega_i^2 + s^2\right) W_i = \sum_i \mu\left(B_i - 2\omega\xi s\right) W_i \rightarrow A_i = \frac{B_i - 2\omega\xi s}{\omega_i^2 + s^2} \qquad (2.16)$$

From (2.6a)

$$\bar{w}\left(x,s\right) = \sum_i \frac{B_i - 2\omega\xi s}{\omega_i^2 + s^2} W_i\left(x\right) \qquad (2.17)$$

but from (2.6b), it is easily deduced that

$$B_i = \int_0^l \bar{P}(\eta, s) W_i\left(\eta\right)\, d\eta \qquad (2.18)$$

So that

$$\bar{w}(x,s)=\sum_i \frac{W_i(x)}{\omega_i^2+s^2}\int_0^l\left[\bar{P}(\eta,s)-2\omega\xi s\right]W_i(\eta)\,d\eta \qquad (2.19)$$

By inversion, the Laplacian transform the solution becomes

$$w(x,t)=\sum_i\int_0^t\int_0^l \frac{W_i(x)W_i(\eta)}{\omega_i}\sin\left[\omega_i(t-\tau)\right]\times P(\eta,\tau)\,d\eta\,d\tau-2\xi\sum_i\int_0^l W_i(x)W_i(\eta)\cos(\omega_i t)\,d\eta \qquad (2.20)$$

The solution turns actual if the convergence of the series involved in (2.20) can be proved. An integration by parts with respect to t is carried out once on the right hand side, and the following is obtained:

$$w(x,t)=\sum_i\int_0^l W_i(x)W_i(\eta)\left\{\frac{P(\eta,t)}{\omega_i^2}-\frac{P(\eta,0)}{\omega_i^2}\cos\omega_i t-\int_0^t\frac{1}{\omega_i^2}\cos\omega_i(t-\tau)\frac{\partial P}{\partial\tau}d\tau-2\xi\cos(\omega_i t)\right\}d\eta \qquad (2.21)$$

Now, let the third term on the right side of (2.21) be investigated on the successively real eigenvalues

$$\left|\sum_i\int_0^l\int_0^t\frac{W_i(x)W_i(\eta)}{\omega_i^2}\cos\omega_i(t-\tau)\frac{\partial P}{\partial\tau}d\tau d\eta\right| \le\sum_i\int_0^l\int_0^t\left|\frac{W_i(x)W_i(\eta)}{\omega_i^2}\right|\left|\cos\omega_i(t-\tau)\right|\left|\frac{\partial P}{\partial\tau}\right|d\tau d\eta \le\int_0^t\left|\frac{\partial P}{\partial\tau}\right|d\tau\sum_i\int_0^l\left|\frac{W_i(x)W_i(\eta)}{\omega_i^2}\right|d\eta \qquad (2.22)$$

Assuming that eigenvalues of the auxiliary problem (2.7) are uniformly bounded in $0\le x\le l$

$$\sum_i\int_0^l\left|\frac{W_i(x)W_i(\eta)}{\omega_i^2}\right|d\eta\le lW^2\sum_i\frac{1}{\omega_i^2}\le\infty. \qquad (2.23)$$

In the case of the last term on the right side in (2.20); since $w(x,0)=0$ is assumed to satisfy the boundary conditions in (2.2), one can expand in terms of the eigenfunctions according to Hilbert's expansion theorem as

$$\sum_i\left\{\int_0^l 2\xi W_i(\eta)\cos(\omega_i t)\,w(\eta,t)\,d\eta\right\}W_i(x) \le\sum_i\left\{\left|\int_0^l 2\xi W_i(\eta)\,w(\eta,t)\,d\eta\right|\left|\cos(\omega_i t)\right|\right\}W_i(x) \le\sum_i\left\{\left|\int_0^l 2\xi W_i(\eta)\,w(\eta,t)\,d\eta\right|\right\}\left|W_i(x)\right|<\mathrm{T}(x,t). \qquad (2.24)$$

In reference to a control functional, T(x,t) in (2.24) is obviously bounded as long as $P_a< P_{a,crit}$, i.e., as long as the eigenvalues are real quantities. Upon introducing the forcing function as

$$P=\left[P_t+P_a w_x(x,t)\right]\delta(x-vt) \qquad (2.25)$$

the magnitude of flexural deflection due to lateral force component can then be estimated as follows:

$$u(x,t)=w_0(x,t)=\int_0^l G(x,\eta;t,t)P_t\delta(\eta-vt)\,d\eta-\int_0^l G(x,\eta;t,0)P_t\delta(\eta-0)\,d\eta-\int_0^l\int_0^t G(x,\eta;t,\tau)\left[P_t\delta(\eta-v\tau)\right]_{,\tau}d\tau d\eta$$
$$=G(x,vt;t,t)P_t-G(x,0;t,0)P_t+\int_0^t G_{,\tau}(x,v\tau;t,\tau)P_t\,d\tau \qquad (2.26)$$

where the Green's function of the integro-differential problem is taken as

$$G(x,\eta;t,\tau)=\sum_i\frac{W_i(x)W_i(\eta)}{\omega_i^2}\cos\omega_i(t-\tau) \qquad (2.27)$$

The final stability and control function upon introducing the axial force component, P_a becomes

$$w(x,t)=u(x,t)+\int_0^t\frac{W_i(x)W_i(v\tau)}{\omega_i}\sin\omega_i(t-\tau)\times P_a w_{0,x}(v\tau,\tau)\,d\tau-T(x,t)$$
$$=\left\{1+\left|\int_0^l 2\xi W_i(\eta)W_i(x)\,d\eta\right|\right\}^{-1}\times\left\{u(x,t)+\int_0^t\frac{W_i(x)W_i(v\tau)}{\omega_i}\times\sin\omega_i(t-\tau)P_a w_{0,x}(v\tau,\tau)\,d\tau\right\} \qquad (2.28)$$

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Stability Condition–
The stability of the inclined beam with respect to the damping ratio is studied as analogous to (2.12) and by a direct variational technique as

$$\det\left|\bar{\Pi}_{jk}\right| = 0;\ \text{and}\ \frac{\partial}{\partial\omega}\left(\det\left|\bar{\Pi}_{jk}\right|\right) = 0 \qquad (2.29)$$

where,

$$\bar{\Pi}_{jk} = \left\{ D\left(\frac{j\pi}{l}\right)^4 + P_a\left(\frac{j\pi}{l}\right)^2 \times \left[\frac{1}{2j\pi}\left(\sin\frac{2j\pi vt}{l} - \sin 2j\pi\right) - \frac{vt}{l}\right] + \left(2\xi_i - 1\right)\mu\omega_i^2 \right\}\delta_{jk} + P_a\left(\frac{j\pi}{l}\right)^2 \times \left[\begin{array}{c}\dfrac{2\sin\pi(j-k) + 2\sin\pi vt/l\,(j-k)}{2\pi(j-k)} - \\ \dfrac{2\sin\pi(j+k) + 2\sin\pi vt/l\,(j+k)}{2\pi(j+k)}\end{array}\right] = 0 \qquad (2.30)$$

such that by simultaneously solving for (2.29) noting (2.30), a solution of critical loads $P_{a,crit}$ are obtained, evident by a set of positive eigenvalues.

III. RESULTS AND DISCUSSION

The following properties are adopted example of a bridge model in [14]: a simply-supported beam of span, L = 25m, modulus of elasticity, E = 2.92GPa, moment of inertia, I = 2.88m^4, mass per unit length, μ = 2351 kg/m. The moving load, P_0 = 50 kN and the speed of the moving load, v = 100km/hr are chosen to be similar to the above reference. Solving (2.13) for j,k ϵ {1,2,…} and applying conditions.

In dimensionless form $\lambda_i^2 = \frac{\mu\omega_i^2 l^4}{D}$ the following eigenvalues are obtained:

$$\lambda_1 = 9.8696,\ \lambda_2 = 39.4784,\ \lambda_3 = 88.8264,\ \lambda_4 = 157.9137,\ \lambda_5 = 246.7401,\ \ldots$$

which showed good agreement with that reported in [14]. The corresponding series of modal vectors can then be represented in the form

$$W_1(\phi,\varsigma) = (-1)\sin\pi(\phi,\varsigma) + (0)\sin 2\pi(\phi,\varsigma) + \ldots$$
$$W_2(\phi,\varsigma) = (0)\sin\pi(\phi,\varsigma) + (-1)\sin 2\pi(\phi,\varsigma) + \ldots$$
$$\vdots \qquad\qquad \vdots$$

Putting: $\phi = \frac{x}{l}$, $\varsigma = \frac{vt}{l}$, $\bar{P}_t = \frac{P_t l^2}{D}$, $\bar{P}_a = \frac{P_a l^2}{D}$, and $0 \le vt \le l$, where t is period of time at which a load P_0 travels within the beam.

Figure 2 analyses the dynamic response of the beam at different load position from the rational of lower bound of eigenvalue for the vibrating system. In (2a-b), the magnitude of deformation, velocity, and acceleration are in good agreement to that obtainable in [14]. Contrary to the analogy stipulated in [14] where so-called 'small' oscillations of vibration exist in the velocity and acceleration of the physical system after five numbers of vibration mode; the evidence of convergence in our case is domicile in just two eigenfunctions (vibration modes).

Figure 3 analyses the dynamic response of the beam at different inclination for undamped and damped (ξ = 0.01) system. The analyses reveal the displacement effect of the inclined beam as it transit in magnitude of force from the lateral component to the axial component of the concentrated load. The results are better described in table 1 and in close agreement to that reported in [14], for a moving force P_0 = 20.18MN (i.e. P = 1.5 in dimensionless form) at motion dependent location ϕ = vt/L. Table 1 further assert the influence of velocity parameter on the displacement magnitude of the beam under different condition of inclination. The displacement due to rise in velocity are seen to be far lower than that liable to influence the stability condition of the beam, contrary to the influence by axial load component.

Table 2 show the minimum critical buckling load obtainable at different angle of beam inclination, and the accompanying set of positive eigenvalues as long as $P_a < P_{a,crit}$, with respect to a damping ratio. The stability for the dynamic system is compared to that obtainable by trial method in [14] by solving (2.29).

It is interesting to observe that the minimum critical loads liable to trigger buckling at any inclination in this approach are reasonably below the critical loads discussed in [14]. The concept of direct variational approach provides a rather exact load capable of immediate distortion of the dynamic system compared to the reference herewith to test the stability. The result agreed that as the inclination reduces, a higher load magnitude will be required to reach buckling in the system. It also revealed that the major stability problem of the dynamic system is conditioned by the magnitude of the axial force component.

Vol. 7, Issue 8, ISSN No. 2455-2143, Pages 102-107
Published Online December 2022 in IJEAST (http://www.ijeast.com)

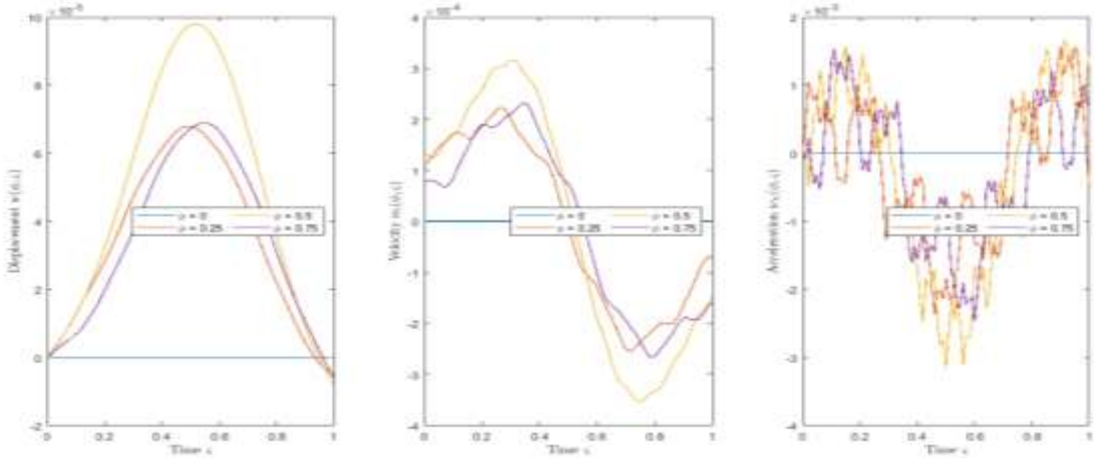

Figure 2. Dynamic response of horizontal beam at load position ϕ = 0, 0.25, 0.5, 0.75 and v = 27.78 m/s (100 km/hr)

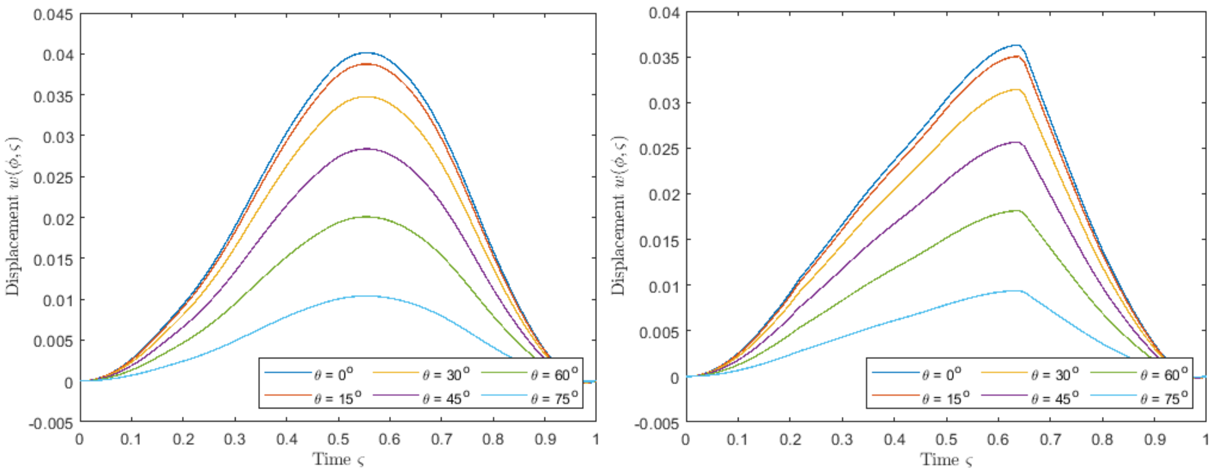

Figure 3. Dynamic response of undamped(left) and damped(right) beam at v = 33.8327 m/s

Table-1. Maximum deflection $w_{max}\left(in\,10^{-2}\right)$ of inclined beam with different speeds

Beam Inclin-ation	v = 3.3833 [S=0.002]		v = 5.3494 [S=0.005]		v = 10.6988 [S=0.02]		v = 33.8327 [S=0. 2]	
	Undamped	Damped	Undamped	Damped	Undamped	Damped	Undamped	Damped
0^o	3.19	2.94[3.14]	3.23	3.35	3.35	3.27[3.18]	4.01	3.63[3.37]
15^o	3.08	2.84[3.09]	3.12	3.24	3.24	3.16[3.13]	3.88	3.50[3.32]
30^o	2.76	2.55[2.82]	2.80	2.90	2.90	2 .83[2.86]	3.48	3.14[3.03]
45^o	2.26	2.08[2.34]	2.29	2.37	2.37	2.31[2.37]	2.84	2.57[2.51]
60^o	1.59	1.47[1.68]	1.62	1.68	1.68	1.63[1.69]	2.01	1.81[1.80]
75^o	0.825	0.761[0.876]	0.837	0.868	0.868	0.846[0.883]	1.04	0.939[0.938]

The values in bracket is that reported in [14].

where,

$$S = \frac{v^2 \mu l^2}{EI} \qquad (3.1)$$

and,

$$\bar{P}_{a,crit} = \frac{\left(P_{a,crit} / \sin\theta\right) l^2}{EI} \qquad (3.2)$$

Table-2. Minimum buckling load at different angles of inclination

θ	$\bar{P}_{cr}$	$\lambda\,(\xi = 0)$	$\lambda\,(\xi = 0.02)$
0^o	∞		
30^o	19.7392		
60^o	11.3964	0, 19.7392	0.0066, 19.7326, 19.7458
80^o	10.0219		
89^o	9.8711 [10.45]		
90^o	9.8696		

The values in bracket is that reported in [14].

IV. CONCLUSION

A close form solution for the moving load of transverse and axial components is dealt with. The resolve by integro-differential approach presented a rather simplistic view in understanding the performance mechanism of the system when compared to other reports. Some findings were highlighted with the aid of examples. The solutions following showed that the magnitude of deflection in beam reduces as the angle of beam inclination increase. The drop in deflection of the beam results in a corresponding decrease in the buckling load required to provide stability of the beam.

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